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Journal of Geometry and Physics 33 (2000) 257–287

JOURNAL OF
GEOMETRY AND
PHYSICS

The geometry of the quantum Euclidean space

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Received 19 July 1999

Abstract

A detailed study is made of the noncommutative geometry of \mathbb{R}_q^3 , the quantum space covariant under the quantum group $SO_q(3)$. For each of its two $SO_q(3)$ -covariant differential calculi we find its metric, the corresponding frame and two torsion-free covariant derivatives that are metric compatible up to a conformal factor and both which yield a vanishing linear curvature. A discussion is given of various ways of imposing reality conditions. The delicate issue of the commutative limit is discussed at the formal algebraic level. Two rather different ways of taking the limit are suggested, yielding $S^2 \times \mathbb{R}$ and \mathbb{R}^3 , respectively, as the limit Riemannian manifolds. ©2000 Elsevier Science B.V. All rights reserved.

Sub. Class.: Quantum groups

MSC: 53C99; 17B37

Keywords: Quantum Euclidean space

1. Introduction and motivation

Already in 1947, Snyder, in an attempt to remove the divergences from electrodynamics, suggested [29] the possibility that the micro-structure of space–time at the Planck level might be better described using a noncommutative geometry. A few years later, in 1954, Pauli suggested that the gravitational field might be considered as a universal regularizer

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for all quantum-field divergencies. An obvious way to reconcile these two points of view is to try to define a gravitational field within the context of noncommutative geometry and see if any connection between the commutation relations and gravity can be found. It is our conjecture that in some yet-to-be found sense the gravitational field is a classical manifestation of the noncommutative micro-structure of space–time. Following the usage in ordinary geometry we shall identify the gravitational field as a metric-compatible, torsion-free linear connection. We shall use the expression ‘connection’ and ‘covariant derivative’ synonymously.

In Section 2 we give a brief overview of a particular version of noncommutative geometry which can be considered as a noncommutative extension of the moving frame formalism of E. Cartan. More details can be found elsewhere [18]. There is not as yet a completely satisfactory definition of either a linear connection or a metric within the context of noncommutative geometry but there are definitions which seem to work in certain cases. In the present article we add to the list of examples which seem to lend weight to one particular definition [8]. We refer to a recent review article [19] for a list of some other examples and references to alternative definitions. More details of one alternative version can be found in the book by Landi [17]; for a general introduction to more mathematical aspects of the subject we refer to the book by Connes [4].

The example we consider here is the 3D ‘quantum Euclidean space’ [10], which has been previously studied by one of the authors. This is the quantum space covariant under the quantum group $SO_q(3)$. In Section 3 we give a review of this space. In Section 4 we recall the construction of the $SO_q(3)$ -covariant differential calculi [1] over them, more specifically the two $SO_q(3)$ -covariant calculi determined in the \hat{R} -matrix formalism [30] and having the de Rham calculus as the commutative limit. More details are to be found in the thesis of one of the authors [11]. We shall find that in order to introduce a ‘frame’ as defined in Section 2 a new generator for dilatation is needed, as in the construction of inhomogenous quantum groups from homogenous ones, as well as the inverse of some generator of the quantum groups from homogenous ones, as well as the inverse of some generator of the quantum Euclidean space. In Section 5 we define a metric and two torsion-free linear connections on the quantum Euclidean space, yielding vanishing linear curvatures. In Section 6 we consider the commutative limit $q \rightarrow 1$ in order to determine the Riemannian manifold which remains as a ‘shadow’ of the noncommutative structure. We suggest two rather different prescriptions for taking the limit: the limit manifold is $S^2 \times \mathbb{R}$ according to the first (and simpler) prescription, \mathbb{R}^3 according to the second (and more sophisticated) one. The initial generators x^i of the quantum space in the former tend to their natural cartesian limits, in the latter are to be ‘renormalized’ before taking the limit. For $q \neq 1$, x^i appear as a noncommutative analog of general (i.e. non-cartesian) coordinates. In Section 7 we discuss the various ways one can construct a real differential calculus from a combination of the two canonical ones. In the last section we present our conclusions and we compare our results with alternative definitions of linear connections. We compare also our results with those found in the case of the Manin plane [6] as well as similar results found [3] in the case of the quantum Euclidean plane of dimension one.

2. Linear connections in the frame formalism

The starting point is a noncommutative algebra \mathcal{A} which has as commutative limit the algebra of functions on some manifold and over \mathcal{A} a differential calculus [4] $\Omega^*(\mathcal{A})$ which has as corresponding limit the ordinary de Rham differential calculus. We recall that a differential calculus is completely determined by the left and right module structure of the \mathcal{A} -module of 1-forms $\Omega^1(\mathcal{A})$. We shall restrict our attention to the case where this module is free of rank n as a left or right module and possesses a special basis θ^a , $1 \leq a \leq n$, which commutes with the elements f of the algebra:

$$[f, \theta^a] = 0. \tag{2.1}$$

In particular the limit manifold must be parallelizable. We shall refer to the θ^a as a frame or stehbein. The integer n plays the role of ‘dimension’; it can be greater than the dimension of the limit manifold but in this case the frame will have a singular limit. We suppose further [6] that the basis is dual to a set of inner derivations $e_a = \text{ad } \lambda_a$. This means that the differential is given by the expression

$$df = e_a f \theta^a = [\lambda_a, f] \theta^a. \tag{2.2}$$

One can rewrite this equation as

$$df = -[\theta, f], \tag{2.3}$$

if one introduces [4] the ‘Dirac operator’

$$\theta = -\lambda_a \theta^a. \tag{2.4}$$

There is a bimodule map π of the space $\Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})$ onto the space $\Omega^2(\mathcal{A})$ of 2-forms and we can write

$$\theta^a \theta^b = P_{cd}^{ab} \theta^c \otimes \theta^d \tag{2.5}$$

where, because of (2.1), the P_{cd}^{ab} belong to the center $\mathcal{Z}(\mathcal{A})$ of \mathcal{A} . We shall suppose that the center is trivial, $\mathcal{Z}(\mathcal{A}) = \mathcal{C}$, and therefore the components P_{cd}^{ab} are real numbers. Define the Maurer–Cartan elements $C_{bc}^a \in \mathcal{A}$ by the equation

$$d\theta^a = \frac{1}{2} C_{bc}^a \theta^b \theta^c. \tag{2.6}$$

Because of (2.5) we can suppose that $C_{bc}^a P_{de}^{bc} = C_{de}^a$. It follows from the equation $d(\theta^a f - f \theta^a) = 0$ that there exist elements F_{bc}^a of the center such that

$$C_{bc}^a = F_{bc}^a - 2\lambda_e P_{bc}^{(ae)} \tag{2.7}$$

where (ab) indicates symmetrization of the indices a and b . If, on the other hand, we define K_{ab} by the equation

$$d\theta + \theta^2 = \frac{1}{2} K_{ab} \theta^a \theta^b, \tag{2.8}$$

then it follows from (2.3) and the identity $d^2 = 0$ that the K_{ab} must belong to the center. Finally it can be shown [6,20] that in order that (2.7) and (2.8) be consistent with one another the original λ_a must satisfy the condition:

$$2\lambda_c \lambda_d P_{ab}^{cd} - \lambda_c F_{ab}^c - K_{ab} = 0. \tag{2.9}$$

This gives to the set of λ_a the structure of a twisted Lie algebra with a central extension.

We propose as definition of a linear connection a map [16]

$$\Omega^1(\mathcal{A}) \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{2.10}$$

which satisfies both a left Leibniz rule

$$D(f\xi) = df \otimes \xi + fD\xi \tag{2.11}$$

and a right Leibniz rule [8]

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f \tag{2.12}$$

for arbitrary $f \in \mathcal{A}$ and $\xi \in \Omega^1(\mathcal{A})$. We have here introduced a generalized flip

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{2.13}$$

in order to define a right Leibniz rule which is consistent with the left one. It is necessarily bilinear. A linear connection is therefore a couple (D, σ) . It can be shown that a necessary as well as sufficient condition for torsion to be right linear is that σ satisfy the consistency condition

$$\pi \circ (\sigma + 1) = 0. \tag{2.14}$$

Using the fact that π is a projection one sees that the most general solution to this equation is given by

$$1 + \sigma = (1 - \pi) \circ \tau, \tag{2.15}$$

where τ is an arbitrary bilinear map

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\tau} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{2.16}$$

If we choose $\tau = 2$ then we find $\sigma = 1 - 2\pi$ and $\sigma^2 = 1$. The eigenvalues of σ are then equal to ± 1 . This general formalism can be applied in particular to differential calculi with a frame. Since $\Omega^1(\mathcal{A})$ is a free module the maps σ and τ can be defined by their action on the basis elements:

$$\sigma(\theta^a \otimes \theta^b) = S_{cd}^{ab} \theta^c \otimes \theta^d, \quad \tau(\theta^a \otimes \theta^b) = T_{cd}^{ab} \theta^c \otimes \theta^d. \tag{2.17}$$

By the sequence of identities

$$f S_{cd}^{ab} \theta^c \otimes \theta^d = \sigma(f\theta^a \otimes \theta^b) = \sigma(\theta^a \otimes \theta^b f) = S_{cd}^{ab} f \theta^c \otimes \theta^d \tag{2.18}$$

and the corresponding ones for T_{cd}^{ab} we conclude that the coefficients S_{cd}^{ab} and T_{cd}^{ab} must lie in $\mathcal{Z}(\mathcal{A})$. From (2.15) the most general form for S_{cd}^{ab} is

$$S_{cd}^{ab} = T_{ef}^{ab}(\delta_c^e \delta_d^f - P_{cd}^{ef}) - \delta_c^a \delta_d^b. \tag{2.19}$$

A covariant derivative can be defined also by its action on the basis elements:

$$D\theta^a = -\omega_{bc}^a \theta^b \otimes \theta^c. \tag{2.20}$$

The coefficients here are elements of the algebra. They are restricted by (2.1) and the two Leibniz rules. The torsion 2-form is defined as usual as

$$\Theta^a = d\theta^a - \pi \circ D\theta^a. \tag{2.21}$$

If $F_{bc}^a = 0$ then it is easy to check [9] that

$$D_{(0)}\theta^a = \theta \otimes \theta^a + \sigma(\theta^a \otimes \theta) \tag{2.22}$$

defines a torsion-free covariant derivative. The most general D for fixed σ is of the form

$$D = D_{(0)} + \chi, \tag{2.23}$$

where χ is an arbitrary bimodule morphism

$$\Omega^1(\mathcal{A}) \xrightarrow{\chi} \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}). \tag{2.24}$$

If we write

$$\chi(\theta^a) = -\chi_{bc}^a \theta^b \otimes \theta^c \tag{2.25}$$

we conclude that $\chi_{bc}^a \in \mathcal{Z}(\mathcal{A})$. In general, a covariant derivative is torsion-free provided the condition

$$\omega_{de}^a P_{bc}^{de} = \frac{1}{2} C_{bc}^a \tag{2.26}$$

is satisfied. The covariant derivative (2.23) is torsion-free if and only if

$$\pi \circ \chi = 0. \tag{2.27}$$

One can define a metric by the condition

$$g(\theta^a \otimes \theta^b) = g^{ab}, \tag{2.28}$$

where the coefficients g^{ab} are elements of \mathcal{A} . To be well defined on all elements of the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ the metric must be bilinear and by the sequence of identities

$$fg^{ab} = g(f\theta^a \otimes \theta^b) = g(\theta^a \otimes \theta^b f) = g^{ab} f \tag{2.29}$$

one concludes that the coefficients must lie in $\mathcal{Z}(\mathcal{A})$. We define the metric to be symmetric if

$$g \circ \sigma \propto g. \tag{2.30}$$

This is a natural generalization of the situation in ordinary differential geometry where symmetry is with respect to the flip which defines the forms. If $g^{ab} = g^{ba}$ then by a linear transformation of the original λ_a one can make g^{ab} the components of the Euclidean (or Minkowski) metric in dimension n . It will not necessarily be symmetric in the sense that we have just used the word.

Introduce the standard notation $\sigma_{12} = \sigma \otimes 1$, $\sigma_{23} = 1 \otimes \sigma$, to extend to three factors of a module any operator σ defined on a tensor product of two factors. Then there is a natural continuation of the map (2.10) to the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ given by the map

$$D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta). \tag{2.31}$$

The map $D_2 \circ D$ has no nice properties but if one introduces the notation $\pi_{12} = \pi \otimes 1$ then by analogy with the commutative case one can set

$$D^2 = \pi_{12} \circ D_2 \circ D \tag{2.32}$$

and formally define the curvature as the map

$$\text{Curv} = D^2. \tag{2.33}$$

Because of the condition (2.14) Curv is left linear. It can be written out in terms of the frame as

$$\text{Curv}(\theta^a) = -\frac{1}{2} R^a_{bcd} \theta^c \theta^d \otimes \theta^b, \tag{2.34}$$

Similarly one can define a Ricci map

$$\text{Ric}(\theta^a) = \frac{1}{2} R^a_{bcd} \theta^c g(\theta^d \otimes \theta^b). \tag{2.35}$$

It is given by

$$\text{Ric}(\theta^a) = R^a_b \theta^b. \tag{2.36}$$

The above definition of curvature is not satisfactory in the noncommutative case. For a discussion of this point we refer to the article by Dubois-Violette et al. [8].

The curvature of the covariant derivative $D_{(0)}$ defined in (2.22) can be readily calculated. One finds after a short calculation that it is given by the expression

$$\begin{aligned} \text{Curv}(\theta^a) &= \theta^2 \otimes \theta^a = \pi_{12} \sigma_{23} \sigma_{12} (\theta^a \otimes \theta \otimes \theta) \\ &= \theta^2 \otimes \theta^a + \pi_{12} \sigma_{12} \sigma_{23} \sigma_{12} (\theta^a \otimes \theta \otimes \theta). \end{aligned} \tag{2.37}$$

If $\xi = \xi_a \theta^a$ is a general 1-form then since Curve is left linear one can write

$$\begin{aligned} \text{Curv}(\xi) &= \xi_a \theta^2 \otimes \theta^a = \pi_{12} \sigma_{23} \sigma_{12} (\xi \otimes \theta \otimes \theta) \\ &= \xi_a \theta^2 \otimes \theta^a + \pi_{12} \sigma_{12} \sigma_{23} \sigma_{12} (\xi \otimes \theta \otimes \theta). \end{aligned} \tag{2.38}$$

We shall use this latter expression in Section 5.

The compatibility of the covariant derivative (2.20) with the metric is expressed by the condition [7].

$$g_{23} \circ D_2 = d \circ g. \tag{2.39}$$

The covariant derivative (2.20) is compatible with the metric if and only if [6].

$$\omega_{bc}^a + \omega_{cd}^e S_{be}^{ad} = 0. \tag{2.40}$$

This is a ‘twisted’ form of the usual condition that $g_{ad}\omega_{bc}^d$ be antisymmetric in the two indices a and c which in turn expresses the fact that for fixed b the ω_{bc}^a form a representation of the Lie algebra of the Euclidean group $SO(N)$ (or the Lorentz group). When $F_{bc}^a = 0$ the condition that (2.22) be metric compatible can be written as

$$S_{df}^{ae} g^{fg} S_{eg}^{bc} = g^{ab} \delta_d^c. \tag{2.41}$$

The algebra we shall consider is a $*$ -algebra and we require that the differential calculus be such that the reality condition

$$(df)^* = df^* \tag{2.42}$$

holds. Neither of the two differential calculi we shall introduce in Section 4 satisfies this condition. In Section 7 we shall discuss how to construct a real calculus $\Omega_R^*(\mathcal{A})$ by taking a subalgebra of the tensor product of the two calculi. We shall require that for arbitrary $f \in \mathcal{A}$ and $\xi \in \Omega_R^1(\mathcal{A})$ one has

$$(f\xi)^* = \xi^* f^*, \quad (\xi f)^* = f^* \xi^*. \tag{2.43}$$

We shall suppose [7] that the extension of the involution to the tensor product is given by

$$(\xi \otimes \eta)^* = \sigma(\eta^* \otimes \xi^*). \tag{2.44}$$

A change in σ therefore implies a change in the definition of an hermitian tensor. The reality condition for the metric becomes then [14]

$$g \circ \sigma(\eta^* \otimes \xi^*) = (g(\xi \otimes \eta))^* \tag{2.45}$$

There is also a reality condition on the covariant derivative and the curvature [14] which implies that the generalized flip σ must satisfy the braid equation.

3. The quantum Euclidean space

The R -matrix or braid matrix $\hat{R} \equiv \|\hat{R}_{ij}^{hk}\|$ of $SO_q(3)$ [10] is a $3^2 \times 3^2$ matrix satisfying the braid equation

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}. \tag{3.1}$$

Here we have used the conventional tensor notation $\hat{R}_{12} = \hat{R} \otimes 1$ used above for σ . By repeated application of Eq. (3.1) one finds

$$f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}f(\hat{R}_{23}) \tag{3.2}$$

for any polynomial function $f(t)$ in one variable. Eqs. (3.1) and (3.2) are evidently satisfied also after the replacement $\hat{R} \rightarrow \hat{R}^{-1}$. The braid matrix admits the projector decomposition

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{-2}\mathcal{P}_t \tag{3.3}$$

with

$$\mathcal{P}_\mu\mathcal{P}_\nu = \mathcal{P}_\mu\delta_{\mu\nu}, \quad \sum_\mu \mathcal{P}_\mu = 1, \quad \mu, \nu = s, t, a. \tag{3.4}$$

The $\mathcal{P}_s, \mathcal{P}_a, \mathcal{P}_t$ are $SO_q(3)$ -covariant q -deformations of respectively the symmetric trace-free, antisymmetric and trace projectors. The trace projector is one-dimensional and its matrix elements can be written in the form

$$\mathcal{P}_t^{ij}{}_{kl} = (g^{mn}g_{mn})^{-1}g^{ij}g_{kl}, \tag{3.5}$$

where [10] the ‘metric matrix’ $g = (g_{ij})$ is a $SO_q(3)$ -isotropic tensor, a deformation of the metric matrix on the classical Euclidean space. The \hat{R} and g satisfy the relations

$$g_{il}\hat{R}_{jk}^{\pm 1lh} = \hat{R}_{ij}^{\mp 1hl}g_{lk}, \quad g^{il}\hat{R}_{lh}^{\pm 1jk} = \hat{R}_{hl}^{\mp 1ij}g^{lk}. \tag{3.6}$$

The lower-case Latin indices i, j, \dots will take the values $(-, 0, +)$. The quantum Euclidean space is the formal (associative) algebra \mathcal{A} with generators $x^i = (x^-, x^0, x^+)$ and relations

$$\mathcal{P}_a^{ij}{}_{kl}x^kx^l = 0 \tag{3.7}$$

for all i, j . We introduce a grading in \mathcal{A} by requiring that the degree of (x^-, x^0, x^+) be equal to $(-1, 0, +1)$, respectively. The matrix elements of \hat{R} and therefore of all the projectors fulfill the condition

$$\hat{R}_{kl}^{ij} = 0 \quad \text{if } i + j \neq k + l. \tag{3.8}$$

Consequently, all the terms appearing on the left-hand side of Eq. (3.7) have the same total degree $i + j$. If we use the explicit expression [10] for \mathcal{P}_a and set

$$h = \sqrt{q} - 1/\sqrt{q}, \tag{3.9}$$

then the relations (3.7) can be written in the form

$$x^-x^0 = qx^0x^-, \quad x^+x^0 = q^{-1}x^0x^+, \quad [x^+, x^-] = h(x^0)^2. \tag{3.10}$$

The first two equations define two copies of the q -quantum plane with q and q^{-1} as deformation parameters and a common generator x^0 .

The metric matrix is given by $g_{ij} = g^{ij}$ with

$$g_{ij} = \begin{pmatrix} 0 & 0 & 1/\sqrt{q} \\ 0 & 1 & 0 \\ \sqrt{q} & 0 & 0 \end{pmatrix}. \tag{3.11}$$

When $q \in \mathbb{R}^+$ one obtains the real quantum Euclidean space by giving \mathcal{A} the structure of a $*$ -algebra with

$$(x^-)^* = \sqrt{q}x^+, \quad (x^0)^* = x^0, \quad (x^+)^* = \frac{1}{\sqrt{q}}x^-. \tag{3.12}$$

This can be written in terms of the metric as

$$(x^i)^* = x^j g_{ji}. \tag{3.13}$$

We can use the summation convention if we consider the involution to lower (or raise) an index. The condition (3.13) is an $SO_q(3, \mathbb{R})$ -covariant condition and three linearly independent, hermitian ‘coordinates’ can be obtained as combinations of the x^i . We define

$$x^r = \Lambda_i^r x^i, \quad \Lambda_i^r := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \sqrt{q} \\ 0 & \sqrt{2} & 0 \\ i & 0 & -i\sqrt{q} \end{pmatrix}. \tag{3.14}$$

With respect to the new ‘coordinates’ the metric is given by

$$g^{rs} = g^{ij} \Lambda_i^r \Lambda_j^s = \frac{1}{2} \begin{pmatrix} q+1 & 0 & i(q-1) \\ 0 & 2 & 0 \\ -i(q-1) & 0 & q+1 \end{pmatrix}.$$

The metric is hermitian but no longer real. In the limit $q \rightarrow 1$ one sees that $g^{rs} \rightarrow \delta^{rs}$. It is more convenient to remain with the original ‘coordinates’ and a real metric.

The ‘length’ squared

$$r^2 := g_{ij} x^i x^j \tag{3.15}$$

is an $SO_q(3, \mathbb{R})$ -invariant, real and generates the center $Z(\mathcal{A}$ of $\mathcal{A})$. Using (3.13) and (3.10) it can be written also in the forms

$$r^2 = (x^i)^* x^i = \left(\sqrt{q} + \frac{1}{q}\right) x^- x^+ + q(x^0)^2 = \left(\sqrt{q} + \frac{1}{\sqrt{q}}\right) x^+ x^- + q^{-1}(x^0)^2. \tag{3.16}$$

The commutation relation between x^+ and x^- can also be written in the form

$$q x^+ x^- - q^- x^- x^+ = h r^2. \tag{3.17}$$

There is obviously no obstruction to extending \mathcal{A} by adding to it the inverse r^{-2} of r^2 , and also the square roots r, r^{-1} of these two elements. We shall further extend the algebra \mathcal{A} by adding the inverse $(x^0)^{-1}$ of x^0 as a new generator with the obvious extra relations.

Finally, we observe that if $|q| = 1$ a compatible involution is defined rather by $(x^i)^* = x^i$. The algebra describes a quantum space covariant under the noncompact section $SO_q(2, 1)$ of $SO_q(3, \mathbb{C})$.

4. The q -deformed Euclidean differential calculi

There are two $SO_q(3)$ -covariant differential calculi [1] $\Omega^*(\mathcal{A})$ and $\bar{\Omega}^*(\mathcal{A})$ over \mathcal{A} neither of which satisfy the reality condition (2.42). In this section we shall write them in a form which will permit us in Section 7 to look for a sensible definition of a real differential calculus. Let d and \bar{d} be the respective differentials and set $\xi^i = dx^i$ and $\bar{\xi}^i = \bar{d}x^i$. The calculi are determined respectively by the commutation relations

$$x^i \xi^j = q \hat{R}_{kl}^{ij} \xi^k x^l \tag{4.1}$$

for $\Omega^1(\mathcal{A})$ and

$$x^i \bar{\xi}^j = q^{-1} \hat{R}_{kl}^{-1ij} \bar{\xi}^k x^l \tag{4.2}$$

for $\bar{\Omega}^1(\mathcal{A})$. Using formulae (3.6) and the fact that we would like to require that the calculus satisfies (2.42), it is easy to see [25] that, if $q \in \mathbb{R}^+$, it is not possible to extend the involution (3.13) to either calculus. This is possible only if $|q| = 1$, and will be considered elsewhere. On the other hand, if $q \in \mathbb{R}^+$ the involution can be extended to the direct sum $\Omega^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A})$ by setting, as well as (2.43),

$$(\xi^i)^* = \bar{\xi}^j g_{ji}, \tag{4.3}$$

since this exchanges relations (4.1) and (4.2). This will be equivalent to

$$(df)^* = \bar{d}f^*. \tag{4.4}$$

If in the limit $q \rightarrow 1$ we identify $\bar{\xi}^i = \xi^i$, we recover the standard involution and the differential is real.

By means of Eqs. (4.1) and (3.6) it is straightforward to check that the commutation relations

$$r^2 \xi^i = q^2 \xi^i r^2, \tag{4.5}$$

are satisfied. The commutation relations between the ξ^i are derived by taking the differential of (4.1):

$$\mathcal{P}_{s_{kl}}^{ij} \xi^k \xi^l = 0, \quad \mathcal{P}_{t_{kl}}^{ij} \xi^k \xi^l = 0. \tag{4.6}$$

If we extend the grading of \mathcal{A} to $\Omega^1 \mathcal{A}$ by setting $\deg(\xi^i) = \deg(x^i)$, we find that each term in (4.1) and (4.6) must have the same total degree.

Written out explicitly Eqs. (4.1) become

$$x^- \xi^- = q^2 \xi^- x^-, \quad x^0 \xi^- = q \xi^- x^0, \quad x^+ \xi^- = \xi^- x^+, \tag{4.7}$$

$$\begin{aligned} x^- \xi^0 &= q \xi^0 x^- + (q^2 - 1) \xi^- x^0, & x^0 \xi^0 &= q \xi^0 x^0 - h(q + 1) \xi^- x^+, \\ x^+ \xi^0 &= q \xi^0 x^+, \end{aligned} \tag{4.8}$$

$$\begin{aligned} x^- \xi^+ &= \xi^+ x^- - h(q + 1) \xi^0 x^0 + h^2(q + 1) \xi^- x^+, \\ x^0 \xi^+ &= q \xi^+ x^0 + (q^2 - 1) \xi^0 x^+, & x^+ \xi^+ &= q^2 \xi^+ x^+ \end{aligned} \tag{4.9}$$

and Eqs. (4.6) become [12]

$$\begin{aligned} (\xi^\pm)^2 &= 0, & (\xi^0)^2 &= h \xi^- \xi^+, & \xi^- \xi^0 &= -q^{-1} \xi^0 \xi^-, & \xi^0 \xi^+ &= -q^{-1} \xi^+ \xi^0, \\ \xi^- \xi^+ &= -\xi^+ \xi^-. \end{aligned} \tag{4.10}$$

Consider the $SO_q(3)$ -invariant 1-form

$$\eta := g_{ij} x^i \xi^j = q^{-1} g_{ij} \xi^j x^i. \tag{4.11}$$

Using the projector decomposition of the \hat{R} -matrix, the relations (3.6) between the \hat{R} -matrix and its inverse as well as the relations (3.7) which define the algebra one can easily verify that

$$[\eta, x^i] = -q^{-2}(q - 1)r^2 \xi^i. \tag{4.12}$$

Hence we conclude, as pointed out in [28,31], that

$$\theta := (q - 1)^{-1} q^2 r^{-2} \eta \tag{4.13}$$

is the ‘Dirac operator’ (2.4) of $\Omega^1(\mathcal{A})$. It satisfies the conditions

$$d\theta = 0, \quad \theta^2 = 0. \tag{4.14}$$

It is of interest to note that

$$dr^2 = (1 - q^{-2})r^2 \theta. \tag{4.15}$$

Since $r \in Z(\mathcal{A})$, from relations (4.15) one concludes immediately that the differential calculus cannot be based on derivatives as outlined in the first section. That is, there can exist no decompositions of θ as in (2.4). This fact is not necessarily a defect but we shall change it anyway by adding an extra element, the ‘dilatator’, to the algebra since at the same time we can reduce $Z(\mathcal{A})$ to \mathbb{C} .

The most general ansatz for the frame can be written in the form $\theta^a := \theta_i^a \xi^i$ where the θ_i^a are elements of \mathcal{A} . We shall let the lower-case Latin indices a, b, \dots take the same values $(+, 0, -)$ as i, j, \dots . The condition (2.1) implies that

$$(r^2 \theta_i^a - q^{-2} \theta_i^a r^2) \xi^i = 0$$

from which we can conclude that

$$r^2 \theta_i^a - q^{-2} \theta_i^a r^2 = 0.$$

This equation has obviously no solution since $r^2 \in Z\mathcal{A}$. To remedy this problem we extend the algebra \mathcal{A} by adding an extra generator Λ , the ‘dilatator’, and its inverse Λ^{-1} , chosen such that

$$x^i \Lambda = q \Lambda x^i. \tag{4.16}$$

The introduction of a new generator Λ is necessary also in a different context, namely in the inhomogenous extension of the homogeneous quantum groups $SL_q(N)$, $SO_q(N)$ and q -Lorentz [21,24,27]; more precisely, Λ appears in the coproduct of the translation part generators. We do not know if this is a coincidence or there is some more fundamental link between the two phenomena.

We extend the original algebra \mathcal{A} defined by Eqs. (3.10) by the addition not only of $(x^0)^{-1}$, $r \pm 1$ but also of $\Lambda^{\pm 1}$ as new generators. Since

$$r \Lambda = q \Lambda r$$

clearly the center of \mathcal{A} is now trivial: $Z\mathcal{A} = \mathbb{C}$. It is natural to extend the grading by setting $\text{deg}(\Lambda) = 0$. We shall choose Λ to be unitary

$$\Lambda^* = \Lambda^{-1}. \tag{4.17}$$

To within a normalization this is the only choice compatible with the commutation relations.

We can now consider the previous ansatz for the frame but with coefficients in the algebra \mathcal{A} . We must assume a linear dependence on the generator Λ and write

$$\theta^a := \Lambda^{-1} \theta_j^a \xi^j, \tag{4.18}$$

where the θ_j^a are elements of \mathcal{A} which do not depend on Λ . The Λ -dependence is here dictated by the condition $[r, \theta^a] = 0$. The condition $[x^i, \theta^a] = 0$ becomes

$$x^i \theta_j^a = \hat{R}_{ij}^{-1ki} \theta_k^a x^l. \tag{4.19}$$

Written out explicitly these equations become

$$\begin{aligned} x^- \theta_-^a &= q^{-1} \theta_-^a x^- - q^{-1} (q^2 - 1) \theta_0^a x^0 + h^2 (1 + q) \theta_+^a x^+, \\ x^0 \theta_-^a &= \theta_-^a = \theta_-^a x^0 + h(1 + q) \theta_0^a x^+, \quad x^+ \theta_-^a = q \theta_-^a x^+, \end{aligned} \tag{4.20}$$

$$\begin{aligned} x^- \theta_0^a &= \theta_0^a x^- + h(1 + q) \theta_+^a x^0, \quad x^0 \theta_0^a = \theta_0^a x^0 - q^{-1} (q^2 - 1) \theta_+^a x^+, \\ x^+ \theta_0^a &= \theta_0^a x^+, \end{aligned} \tag{4.21}$$

$$x^- \theta_+^a = q \theta_+^a x^-, \quad x^0 \theta_+^a = \theta_+^a x^0, \quad x^+ \theta_+^a = q^{-1} \theta_+^a x^+. \tag{4.22}$$

Eqs. (4.22) contain only θ_+^a and admit, to within an arbitrary factor in the center of \mathcal{A} , only two independent solutions. We choose

$$\theta_+^- = 0, \quad \theta_+^0 = 0, \quad \theta_+^+ = r^{-2} x^0,$$

so that θ^+ contains a term proportional to ξ^+ . We replace this result in Eq. (4.21) which become then equations for θ_0^a and we find the solutions

$$\theta_0^- = 0, \quad \theta_0^0 = r^{-1}, \quad \theta_0^+ = -(q + 1)r^{-2}x^+.$$

Finally, Eqs. (4.20) can be solved for θ_-^a yielding

$$\begin{aligned} \theta_-^- &= (x^0)^{-1}, & \theta_-^0 &= \sqrt{q}(q + 1)r^{-1}(x^0)^{-1}x^+, \\ \theta_-^+ &= -\sqrt{q}q(q + 1)r^{-2}(x^0)^{-1}(x^+)^2. \end{aligned}$$

If we place these coefficients into Eq. (4.18) we completely determine the frame. We shall include a rescaling by a normalization factor α , which we leave free for the moment:

$$\begin{aligned} \theta^- &= \alpha^{-1}\Lambda^{-1}(x^0)^{-1}\xi^-, & \theta^0 &= \alpha^{-1}\Lambda^{-1}r^{-1}(\sqrt{q}(q + 1)(x^0)^{-1}x^+\xi^- + \xi^0), \\ \theta^+ &= \alpha^{-1}\Lambda^{-1}r^{-2}(-\sqrt{q}q(q + 1)(x^0)^{-1}(x^+)^2\xi^- - (q + 1)x^+\xi^0 + x^0\xi^+). \end{aligned} \quad (4.23)$$

Note that we have enumerated the θ^a so that $\text{deg}(\theta^a) = (-1, 0, 1)$, exactly as for the x^i and ξ^i .

The above θ^a are determined up to linear transformations with coefficients depending on r . The commutation rules between Λ and θ^a will depend on their r -normalization as well as the commutation rules between Λ and ξ^i , which we have not specified yet. By differentiating Eq. (4.16) we obtain the condition

$$\xi^i\Lambda + x^i d\Lambda = q d\Lambda x^i + q\Lambda\xi^i \quad (4.24)$$

on the differential $d\Lambda$. A possible solution is given by the two conditions

$$x^i d\Lambda = q d\Lambda x^i, \quad \xi^i\Lambda = q\Lambda\xi^i. \quad (4.25)$$

In particular, one can consistently set $d\Lambda = 0$. We shall do in the sequel although it means that the condition $df = 0$ does not imply that $f \in \mathbb{C}$. This is not entirely satisfactory since one would like the only ‘function’ with vanishing exterior derivative to be the ‘constant function’. It could be remedied by considering a more general solution to Eq. (4.25). This would, however, complicate our calculations since it would increase the number of independent forms by one. A necessary condition for $d\Lambda = 0$ is that $[\Lambda, \lambda_a] = 0$. The λ_a which we give below in Eq. (4.32) satisfy the latter because they are homogeneous functions of x^i of degree zero. The condition that Λ commutes with the θ^a fixes the normalization factor α in \mathbb{R}^+ , first as a fixed scale and then as a suitable function of h , so as to adjust the commutative limit of the frame. Another possibility would be to impose a commutation relation between ξ and Λ of the type

$$\Lambda\xi^i = \xi^i\Lambda. \quad (4.26)$$

This follows, for example, from the condition [23] $\Lambda df = qd\Lambda f$ which in turn would be equivalent to considering Λ a suitable element of the associated Heisenberg algebra (in fact, such an element has a simple realization in terms of x^i and $SO_q(3)$ -covariant twisted derivatives [23]). The condition that Λ commutes with the θ^a would now fix the

normalization factor α in (4.23) to be proportional to r^{-1} and will thus change the metric by a conformal factor r^2 . After imposing either commutation rule between Λ and θ^a the frame will be determined up to a linear transformation with coefficients in \mathbb{C} .

A direct and lengthy calculation (performed using the program for symbolic computations REDUCE) with the explicit expression for the matrix \hat{R}_{ij}^{ab} [10] shows that the θ_i^a satisfy the relations

$$\hat{R}_{cd}^{ab} \theta_j^d \theta_i^c = \theta_i^b \theta_k^a \hat{R}_{ij}^{kl}. \tag{4.27}$$

There are $3^4 = 81$ equations. However, since both $R_{ij}^{kl} \equiv \hat{R}_{ij}^{lk}$ and θ_i^a are lower triangular matrices, $3^2(3^2 - 1)/2 = 36$ of them are trivial identities. The proof actually consists then in checking ‘only’ 45 equations. By repeated application of relations (4.27) it immediately follows that the same relations hold also after replacing \hat{R} by any polynomial $f(\hat{R})$,

$$f(\hat{R})_{cd}^{ab} \theta_j^d \theta_i^c = \theta_i^b \theta_k^a f(\hat{R})_{ij}^{kl}. \tag{4.28}$$

In particular we can choose $f(\hat{R}) = \mathcal{P}_t, \mathcal{P}_s$. With the help of relations (2.1) and (4.18) we find as a consequence that the θ^a have the same commutation relations as the ξ^i :

$$\mathcal{P}_{cd}^{ab} \theta^c \theta^d = 0 \quad \mathcal{P}_{scd}^{ab} \theta^c \theta^d = 0. \tag{4.29}$$

Therefore the P of Eq. (2.9) is given by

$$P = \mathcal{P}_a. \tag{4.30}$$

It is the $SO_q(3)$ -covariant deformation of the antisymmetric projector. Note also that for $f(\hat{R}) = \mathcal{P}_t$ Eq. (4.28) is equivalent to the relations

$$g_{cd} \theta_j^d \theta_i^c = \kappa g_{ij}, \quad g^{ij} \theta_j^b \theta_i^a = \kappa g^{ab} \tag{4.31}$$

with

$$\kappa = (gklg^{kl})^{-1} g_{ab} \theta_i^b \theta_j^a g^{ji} = r^{-2} \alpha^{-2}$$

because of Eq. (3.5).

Consider the elements $\lambda_a \in \mathcal{A}$ with

$$\lambda_- = h^{-1} q \Lambda (x^0)^{-1} \alpha x^+, \quad \lambda_0 = -h - 1\sqrt{q} \Lambda \alpha (x^0)^{-1} r, \quad \lambda_+ = -h^{-1} \Lambda \alpha (x^0)^{-1} x^-. \tag{4.32}$$

By direct calculation one verifies that $\theta = -\lambda_a \theta^a$ is given by (4.13). Since Λ is unitary the hermitian adjoints λ_a^* are given by

$$\lambda_{\pm}^* = -\Lambda^{-2} g^{\pm b} \lambda_b, \quad \lambda_0^* = \Lambda^{-2} g^{0b} \lambda_b. \tag{4.33}$$

The fact that the λ_a are not anti-hermitian is related to the fact that the differential d is not real. We have chosen this rather odd normalization to have the commutation relations (4.38) below. A straightforward calculation yields the commutation relations $[\lambda_a, x^i] = q \Lambda e_a^i$ with

$$(e^i_a) = \alpha \begin{pmatrix} +x^0 & 0 & 0 \\ -(\sqrt{q} + 1/\sqrt{q})x^+ & r & 0 \\ -\sqrt{q}(q + 1)(x^0)^{-1}(x^+)^2 & (q + 1)r(x^0)^{-1}x^+ & r^2(x^0)^{-1} \end{pmatrix}. \quad (4.34)$$

The e^i_a is the inverse matrix of θ^a_i :

$$e^i_a \theta^a_j = \delta^i_j, \quad \theta^a_j e^j_b = \delta^a_b. \quad (4.35)$$

Relations (4.27) imply that the matrix elements e^i_a satisfy also the ‘RTT-relations’ [10]

$$\hat{R}^{ij}_kl e^k_a e^l_b = e^i_c e^j_d \hat{R}^{cd}_{ab} \quad (4.36)$$

as well as the ‘gTT-relations’ [10]

$$g^{ab} e^i_a e^j_b = r^2 \alpha^2 g^{ij}, \quad g_{ij} e^i_a e^j_b = r^2 \alpha^2 g_{ab}. \quad (4.37)$$

Thus $(\alpha r)^{-1} e^i_a$ fulfill the same commutation relations of the generators T^i_a of $SO_q(3)$ and in this sense may be seen as ‘local’ realization of T^i_a . However, they do not satisfy the same $*$ -relations, nor is there an analog for the coproduct of $SO_q(3)$. The λ_α satisfy the commutation relations

$$\lambda_- \lambda_0 = q \lambda_0 \lambda_-, \quad \lambda_+ \lambda_0 = q^{-1} \lambda_0 \lambda_+, \quad [\lambda_+, \lambda_-] = h(\lambda_0)^2. \quad (4.38)$$

These are the same commutation relations as those satisfied by the x^i . This is a remarkable fact and it underlines how weak a constraint the commutation relations are on the algebra. The λ_a are related in fact to the x^i by a rather complicated nonlinear relation (4.32). They differ, however, from the x^a in that they commute with Λ .

Eq. (4.38) can be rewritten more compactly in the form

$$\mathcal{P}^{ab}_{acd} \lambda_a \lambda_b = 0, \quad (4.39)$$

so that the constants F^a_{bc} and K_{ab} in Eq. (2.9) vanish. Hence Eq. (2.6) is satisfied with

$$C^{abc} = -2\lambda_d \mathcal{P}_a (da)_{bc} \quad (4.40)$$

an expression which is consistent with (2.7) because of (4.30). Finally, it is easy to check that

$$g^{ab} \lambda_a \lambda_b = q h^{-2} (\Lambda \alpha)^2. \quad (4.41)$$

One can easily find the relation between the $SO_q(3)$ -covariant derivatives ∂_i introduced in Ref. [1], which fulfill the modified Leibniz rule

$$\partial_i x^j = \delta^j_i + q \hat{R}^{jh}_{ik} x^k \partial_h, \quad (4.42)$$

and the e_a . From the decomposition $d = \xi^i \partial_i = \theta^a e_a$ it is evident that

$$\partial_i = \Lambda^{-1} \tilde{\theta}^a_i e_a, \quad (4.43)$$

where we have now decomposed θ^a in the form $\theta^a = \xi^i \Lambda^{-1} \tilde{\theta}^a_i$.

We conclude this section by listing the basic formulae which, besides (4.2), characterize the barred differential calculus $\bar{\Omega}^1(\mathcal{A})$. The analogs of (4.6) are obtained by taking the differential of (4.2):

$$\mathcal{P}_{skl}^{ij} \bar{\xi}^k \bar{\xi}^l = 0, \quad \mathcal{P}_{tkl}^{ij} \bar{\xi}^k \bar{\xi}^l = 0. \tag{4.44}$$

All relations are compatible with the grading of \mathcal{A} extended by setting $\text{deg}(\bar{\xi}^i) = \text{deg}(x^i)$. The $SO_q(3)$ -invariant 1-form

$$\bar{\eta} := g_{ij} x^i \bar{\xi}^j = q g_{ij} \bar{\xi}^j x^i \tag{4.45}$$

fulfills in $\bar{\Omega}^1(\mathcal{A})$ the commutation relations

$$[\bar{\eta}, x^i] = -q^2(q^{-1} - 1)r^2 \bar{\xi}^i. \tag{4.46}$$

Hence

$$\bar{\theta} := (q^{-1} - 1)^{-1} q^{-2} r^{-2} \bar{\eta} \tag{4.47}$$

is the ‘Dirac operator’ (2.4) of $\bar{\Omega}^1(\mathcal{A})$, and

$$\bar{d}\bar{\theta} = 0, \quad \bar{\theta}^2 = 0. \tag{4.48}$$

From the definitions of θ , $\bar{\theta}$ and the involution it follows that in $\Omega^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A})$

$$\theta^* = -\bar{\theta}, \quad (\bar{\theta})^* = -\theta. \tag{4.49}$$

It is straightforward to show that

$$\bar{d}r^2 = (1 - q^2)r^2 \bar{\theta}. \tag{4.50}$$

There exist a frame $\bar{\theta}^a$ for the differential calculus $\bar{\Omega}^*(\mathcal{A})$ in the form

$$\bar{\theta}^a := \Lambda \bar{\theta}_j^a \bar{\xi}^j, \tag{4.51}$$

where the $\bar{\theta}_j^a$ are elements of \mathcal{A} which do not depend on Λ . The Λ -dependence is here again dictated by the condition $[r, \bar{\theta}^a] = 0$. The condition $[x^i, \bar{\theta}^a] = 0$ becomes

$$\bar{\theta}_j^a x^j = \hat{R}_{il}^{jk} x^l \bar{\theta}_k^a. \tag{4.52}$$

The elements $\bar{\lambda}_a \in \mathcal{A}$ are introduced through the decomposition

$$\bar{\theta} = -\bar{\lambda}_a \bar{\theta}^a \tag{4.53}$$

and are dual to $\bar{\theta}^a$. As above, $\bar{\theta}^a$ and the corresponding $\bar{\lambda}^a$ are determined up to a linear transformation with coefficients in \mathbb{C} . Now note that from $0 = [\mathcal{A}, \theta^a]^* = [(\theta^a)^*, \mathcal{A}]$ it follows that for q real positive $(\theta^a)^*$ is a combination of $\bar{\theta}^b$. We choose the second basis so that

$$(\theta^a)^* = \bar{\theta}^b g_{ba}. \tag{4.54}$$

This will automatically yield $\text{deg}(\bar{\theta}^a) = (-1, 0, 1)$ and the same commutation relations as for the θ^a , because of relations (3.6):

$$\mathcal{P}_{cd}^{ab} \bar{\theta}^c \bar{\theta}^d = 0, \quad \mathcal{P}_{s cd}^{ab} \bar{\theta}^c \bar{\theta}^d = 0. \tag{4.55}$$

The explicit expressions for $\bar{\theta}^a$ become

$$\begin{aligned} \bar{\theta}^- &= (\alpha q)^{-1} \Lambda r^{-2} \left(x^0 \bar{\xi}^- - (q^{-1} + 1)x^{-} \bar{\xi}^0 - \frac{1}{\sqrt{q}} q^{-1} (q^{-1} + 1)(x^0)^{-1} (x^-)^2 \bar{\xi}^+ \right), \\ \bar{\theta}^0 &= (\alpha q)^{-1} \Lambda r^{-1} \left(\bar{\xi}^0 + \frac{1}{\sqrt{q}} (q^{-1} + 1)(x^0)^{-1} x^{-} \bar{\xi}^+ \right), \quad \bar{\theta}^+ = (\alpha q)^{-1} \Lambda (x^0)^{-1} \bar{\xi}^+. \end{aligned} \tag{4.56}$$

The corresponding $\bar{\lambda}_a$ are given by

$$\bar{\lambda}_\pm = \Lambda^{-2} \lambda_\pm, \quad \bar{\lambda}_0 = -\Lambda^{-2} \lambda^0$$

and therefore the involution on the λ_a becomes

$$\lambda_a^* = -g^{ab} \bar{\lambda}_b. \tag{4.57}$$

This is to be compared with (3.13).

5. Metrics and linear connections on the quantum Euclidean space

We now look for metrics, generalized permutations and covariant derivatives corresponding to each of the above differential calculi. They will be essential ingredients to determine the correct correspondence between mathematical objects and physical observables. We shall see that it is not possible to satisfy all the requirements of Section 2. Nevertheless, leaving the problem of reality aside for the moment, to be treated in Section 7, we show that if we allow a conformal factor in the metric then a unique linear connection exists which is metric compatible and automatically $SO_q(3)$ -covariant.

We define covariant derivatives D on $\Omega^1(\mathcal{A})$ and \bar{D} on $\bar{\Omega}^1(\mathcal{A})$ as maps

$$\begin{aligned} \Omega^1(\mathcal{A}) &\xrightarrow{D} \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}), \\ \bar{\Omega}^1(\mathcal{A}) &\xrightarrow{\bar{D}} \bar{\Omega}^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A}) \end{aligned}$$

which satisfy left and right Leibniz rules.

In accord with Eq. (2.16), we look for a generalized permutation σ by starting with the ansatz

$$\sigma(\theta^a \otimes \theta^b) = S_{cd}^{ab} \theta^c \otimes \theta^d \tag{5.1}$$

with S_{cd}^{ab} complex numbers and we impose the condition (2.14). Eq. (4.29) implies that

$$S = C_s \mathcal{P}_s - \mathcal{P}_a + C_t \mathcal{P}_t, \tag{5.2}$$

where C_s and C_t are complex $3^2 \times 3^2$ matrices. Similarly, according to (2.28) to define a metric g we start with the ansatz

$$g(\theta^a \otimes \theta^b) = g^{ab} \tag{5.3}$$

with g^{ab} again complex numbers. In view of Eq. (3.6) a necessary condition for the metric-compatibility condition (2.41) is that g_{ab} be proportional to the matrix defined in (3.11) and either $S = \hat{R}$ or $S = \hat{R}^{-1}$. Without loss of generality we can set the proportionality factor equal to 1, since this amounts to a redefinition of the factor α in (4.23). But because of (3.3) these forms for S are clearly not compatible with (5.2).

The best one can do is to weaken (2.41) to a condition of proportionality. To fulfill the latter it is sufficient that S be proportional to \hat{R} or \hat{R}^{-1} . Then the double requirement admits the two solutions

$$S = q\hat{R}, \quad S = (q\hat{R})^{-1}, \tag{5.4}$$

corresponding respectively to $C_s = q^2, C_t = q^{-1}$ or $C_s = q^{-2}, C_t = q$. Therefore we have

$$S_{df}^{ae} g^{fg} S_{eg}^{cb} = q^{\pm 2} g^{ac} \delta_d^b. \tag{5.5}$$

In both cases σ fulfills the braid Eq. (3.1). If we compare the above equation with (2.41) we see that the metric (5.3) is not compatible with the linear connection. However, we shall show below in Eq.(6.2) that with a conformal factor it is so; the (flat) linear connection is equal to the Levi–Civita connection of a (flat) metric conformally equivalent to the one which we have found. The symmetry condition (2.30) for the metric follows from (3.6) and from (3.3). Since the matrix g^{ab} is not symmetric in the ordinary sense there can exist no linear transformation $\theta'^a = \Lambda_b^a \theta^b$. Such that $g'^{ab} = \Delta^ab$.

From Eqs. (4.18) and (4.28), and (5.1) it is possible to determine the action of σ and g on the basis ξ^i . One finds that

$$\sigma(\xi^i \otimes \xi^j) = S_{hk}^{ij} \xi^h \otimes \xi^k, \tag{5.5}$$

$$g(\xi^i \otimes \xi^j) = g^{ij} \alpha^2 q^{-1} r^2 \Lambda^2. \tag{5.6}$$

Eq. (5.5) is the same formula as (5.1); Eq. (5.6) is to be compared with Eq. (5.3). From these results it is manifest that σ and g are $SO_q(3)$ -covariant; under the $SO_q(3)$ coaction $\sigma(\xi^i \otimes \xi^j)$ and $g(\xi^i \otimes \xi^j)$ transform as $\xi^i \otimes \xi^j$. Relations (5.5) and (5.6) of course could have been obtained also by a direct calculation in the ξ^i basis.

For either choice of σ the covariant derivative

$$D_{(0)}\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta), \tag{5.7}$$

defined in (2.22) is manifestly $SO_q(3)$ -invariant, because θ is. The most general D is given by Eq. (2.23). But because of Eq. (4.29) we have

$$\pi \circ \chi(\theta^a) = -\chi_{ab}^a \theta^b \theta^c - \chi_{bc}^a (\mathcal{P}_s + \mathcal{P}_a + \mathcal{P}_t)_{dc}^{bc} \theta^d \theta^e = -\chi_{a_{dc}}^{abc} \mathcal{P} \theta^e$$

and because of Eq. (4.30) this extra term must vanish if we require the torsion to vanish.

Finally, one can show that a term $\chi_{bc}^a (\mathcal{P}_s + \mathcal{P}_t)^{bc} \theta^d \otimes \theta^e$ is forbidden by $SO_q(3)$ -covariance. The most general torsion-free and $SO_q(3)$ -covariant linear connection is then given by

$$D = D_{(0)}. \tag{5.8}$$

The linear curvature tensor is given by Eq. (2.38); the first term is absent, because $\theta^2 = 0$. The result that the curvature map Curv is left linear but in general not right linear is particularly evident from this formula. In fact it is easy to see that $\text{Curv} = 0$. This relies only on the fact that S fulfills the braid Eq. (3.1):

$$\begin{aligned} \text{Curv}(\xi) &= \pi_{12} \sigma_{012} \sigma_{023} \sigma_{012} (\theta^a \otimes \theta^b \otimes \theta^c) \xi_a \lambda_b \lambda_c \\ &= \pi_{12} (S_{12} S_{23} S_{12})_{def}^{abc} (\theta^d \otimes \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c \\ &= (S_{12} S_{23} S_{12})_{def}^{abc} (\theta^d \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c = - (S_{12} S_{23} S_{12})_{def}^{abc} (\theta^d \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c \\ &= - (\mathcal{P}_{a23} S_{12} S_{23})_{def}^{abc} (\theta^d \theta^e \otimes \theta^f) \xi_a \lambda_n \lambda_c = 0. \end{aligned}$$

The last three equalities follow from respectively Eq. (2.14), (3.2) and (4.39).

It is interesting to compute the result of the two flat connections on the differentials. On the one hand, if we choose $S = (q\hat{R})^{-1}$ then using formulae (4.1) and (3.6) it is straightforward to show that

$$D\xi^i = 0. \tag{5.9}$$

This means that the ‘coordinates’ are adapted to the zero-curvature condition. On the other hand, it is straightforward to show that if $S = q\hat{R}$ then

$$\begin{aligned} D\xi^i &= (q^2 - 1)\theta \otimes \xi^i - (1 + q^{-1})r^{-2}x^i [q^2 \xi^l \otimes \xi^m g_{lm} - q^4 g_{hj} \hat{R}^{ji} x^h \xi^l \otimes \xi^m] \\ &= (q^2 - 1)[\theta \otimes \xi^i + \xi^i \otimes \theta] - q^2(1 + q^{-1})r^{-2}x^i \xi^l \otimes \xi^m g_{lm}. \end{aligned} \tag{5.10}$$

This is different from zero. In this case the ‘coordinates’ are not well adapted.

We can repeat the same construction for the barred calculus $\bar{\Gamma}^1(\mathcal{A})$. We simply list the result. There are essentially a unique metric g and two generalized permutations σ which fulfill the metric compatibility weakened by a conformal factor. They are defined by

$$g(\bar{\theta}^a \otimes \bar{\theta}^b) = g^{ab}, \tag{5.11}$$

$$\sigma(\bar{\theta}^a \otimes \bar{\theta}^b) = \bar{S}_{cd}^{ab} \bar{\theta}^c \otimes \bar{\theta}^d \tag{5.12}$$

with either

$$\bar{S} = q\hat{R}, \quad \text{or} \quad \bar{S} = (q\hat{R})^{-1}. \tag{5.13}$$

Correspondingly

$$\bar{S}_{df}^{ae} g^{fg} \bar{S}_{eg}^{cb} = q^{\pm 2} g^{ac} \delta_d^b.$$

In the $\bar{\xi}^i$ basis,

$$\sigma(\bar{\xi}^i \otimes \bar{\xi}^j) = \bar{S}_{hk}^{ij} \bar{\xi}^h \otimes \bar{\xi}^k, \tag{5.14}$$

$$g(\bar{\xi}^i \otimes \bar{\xi}^j) = g^{ij} \alpha^2 q r^2 \Lambda^{-2}. \tag{5.15}$$

The most general torsion-free $SO_q(3)$ -covariant and (up to a conformal factor) metric compatible covariant derivatives are

$$\bar{D}\bar{\xi} = -\bar{\theta} \otimes \bar{\xi} + \sigma(\bar{\xi} \otimes \bar{\theta}) \tag{5.16}$$

with either choice of σ . They are manifestly $SO_q(3)$ -invariant. The associated linear curvatures $\overline{\text{Curv}}$ vanish.

If one extends the involution to the second tensor power of the 1-form algebra using (2.44), then it is easy to verify that neither D nor \bar{D} are real. If $\bar{S} = S^{-1}$, they fulfill however instead the relation

$$(D\xi)^* = \bar{D}\xi^*. \tag{5.17}$$

6. The commutative limit

The set (x^i, e_a) generates the algebra \mathcal{D}_h of observables of phase space; the x^i generate the subalgebra of position observables, whereas the e_a the subalgebra of momenta observables. One has to look for a complete set of three (the dimension of the classical underlying manifold) independent commuting observables within the whole \mathcal{D}_h , and not just within the position space subalgebra, since the latter is not abelian. Of course, this is exactly what makes noncommutative geometry interesting, in that it makes completely localized states impossible. Here we shall restrict our consideration especially to position space observables. It should be the theory which indicates what is the correct identification of the operators x^i in commutative limit. By theory we mean the algebra, that is, the theory proper, as well as the representation or the state. Below we find two possible and rather different identifications of the x^i . A physically satisfactory understanding should involve a Hilbert space representation theory of the algebra such that the spectrum of the position observables becomes ‘dense’ in the limit manifold. Since the present analysis is meant to be preliminary and heuristic in character, we shall restrict our attention to the algebra.

In the commutative limit Λ commutes both with coordinates and derivations, and therefore must be equal to a constant. Since we have already normalized Λ so as to be unitary, the constant must be a pure phase. One can always absorb the latter in the same normalization, so that

$$\lim_{q \rightarrow 1} \Lambda = 1. \tag{6.1}$$

In the first identification we pick a fixed real α in Eq. (4.23) and suppose that the generators of the algebra tend to their native natural limit as (complex) coordinates on a real manifold and that the frame tends to the corresponding limit of a moving frame on this manifold. From (5.6) we find that in the commutative limit the metric is given by the line element

$$ds^2 = (\alpha r)^{-2} \delta_{i,-j} dx^i dx^j = (\alpha r)^{-2} (dx^2 + dy^2 + dz^2). \tag{6.2}$$

The second expression is in terms of the real coordinates (3.14). If one uses spherical polar coordinates then one sees immediately that the Riemannian space is $S^2 \times \mathbb{R}$ with

$\log(\alpha r)$ the preferred coordinate along the line. The radius of the sphere is equal to α^{-1} . An interesting feature of this identification is that neither the covariant derivative (5.8) nor (5.16) is compatible with this non-flat metric. This was of course to be expected since none of the σ we have used satisfies the metric compatibility condition (2.41). The problem we are considering here lies in fact a little outside the range of the general theory of Section 2 because of the element Λ which is not in the center but which has nevertheless a vanishing differential. Alternatively if α is proportional rather to r^{-1} the conformal factor in (6.2) would disappear and the metric would be the ordinary flat metric of \mathbb{R}^3 . We observed in Section 4 that this would follow from the commutation relation (4.26). In the commutative limit the frame (4.23) becomes a moving frame in the sense of Cartan. The singularity at $x^0 = 0$ is to be expected, since there can be no global frame on a non-parallelizable manifold like a sphere; for positive and negative values of x^0 one has two different local sections of the frame bundle on the two charts corresponding, respectively, to the upper and lower hemisphere. According to (4.54), the frame is however not real. To find a real frame we first try the same linear transformation (3.14) used for the coordinates

$$\theta^1 = \frac{1}{\sqrt{2}}(\theta^- + \theta^+), \quad \theta^2 = \theta^0, \quad \theta^3 = \frac{i}{\sqrt{2}}(\theta^- - \theta^+).$$

A short calculation yields

$$\begin{aligned} \theta^1 &= (\alpha x^0 r) - 1(r dx - x dr + iz dr), & \theta^2 &= (\alpha x^0 r) - 1(r dr - ix dz + iz dx), \\ \theta^3 &= (\alpha x^0 r) - 1(r dz - ix dr + z dr). \end{aligned} \tag{6.3}$$

Although in the commutative limit the differential is real, we see that the frame is not. Equivalently, from (4.54) we see that the frames for the two differential calculi do not coincide in the commutative limit, although the two differential calculi themselves do. It is often found in specific calculations in General Relativity that it is more convenient to use a complex frame to calculate real curvature invariants. We can see however no property of the frame (6.3) which makes it particularly adapted to study the space $S^2 \times \mathbb{R}$, or the space \mathbb{R}^3 in the case that $\alpha \propto r^{-1}$. A more complete analysis should involve at this point the study of the $*$ -representations of the algebra of observables \mathcal{D}_h , studied e.g. in Ref. [13]. We shall not enter it here, but simply recall that the definition of the commutative limit of these representations is rather delicate.

We shall now propose a different identification of the x^i in the commutative limit. The limit manifold is the desired \mathbb{R}^3 and for finite h the x^i will be a sort of general coordinates on \mathbb{R}^3 . They are related to cartesian coordinates by a transformation which becomes singular in the commutative limit. The singularity can be removed by performing a renormalization procedure [with diverging renormalization constant α^{-1} , or with a function $\alpha^{-1} = \alpha' r$, with α' a diverging constant, in the case (4.26)] before taking the limit. We thus obtain cartesian coordinates y^i on \mathbb{R}^3 . We try to solve algebraically the problems arising from the first identification. We reconsider the search of a real nondegenerate frame in the commutative limit. If the condition $\lim_{h \rightarrow 0} x = 0 = \lim_{h \rightarrow 0} z$, or equivalently

$$x_{(0)}^\pm := \lim_{h \rightarrow 0} x^\pm = 0, \tag{6.4}$$

were fulfilled on all points of the classical manifold, the frame (6.3) would be real in the limit; apparently the latter would also be degenerate ($\theta^1 = 0 = \theta^3$), but the latter consequence can be avoided by choosing a diverging normalization factor α^{-1} in (4.23) and related definitions.

Let us first show by a one-dimensional example how such a renormalization may change the situation. In the sequel we shall append a suffix (0) to any quantity to denote its commutative limit, e.g., $r_{(0)} = \lim_{h \rightarrow 0} r$. Let $y \in \mathbb{R}$ and consider the change of coordinates

$$x = f(\alpha y) \tag{6.5}$$

with some α going to zero when h does and f invertible, which we shall assume normalized in such a way that $f'(0) = 1$. Although $x_{(0)} = f(0) = \text{const.}$, e.g., $f(0) = 0$ we have

$$dx_{(0)} = 0, \quad \left. \frac{\partial}{\partial x} \right|_{h=0} = \infty.$$

Nevertheless, one can extract the original nontrivial coordinate, differential and derivative by taking a difference and performing a rescaling before taking the limit:

$$y = \lim_{h \rightarrow 0} \frac{x - x_{(0)}}{\alpha}, \quad \theta^1 \equiv dy = \lim_{h \rightarrow 0} \frac{dx - dx_{(0)}}{\alpha}, \quad \frac{\partial}{\partial y} = \lim_{h \rightarrow 0} \alpha \frac{\partial}{\partial x}. \tag{6.6}$$

Note that the above convergences are not uniform in y , namely are slower as y gets larger. This example suggests that we may solve the degeneracy problem by taking the normalization factor α in formula (4.23) as a suitable infinitesimal in h .

Now we return to the algebra which interests us here. We aim to show that, upon assuming (6.4), and by choosing an infinitesimal α such the commutative limit of $\alpha^{-1}x^\pm$ is finite, the commutation limit of $\alpha^{-1}x^-, \alpha^{-1}(x^0 - 1), \alpha^{-1}x^+$ is in fact a set of (complex) cartesian coordinates y^-, y^0, y^+ of \mathbb{R}^3 . In addition, we shall also sketch some basic ideas for a correspondence principle between the deformed and the undeformed theory for finite h . As a first step, we show that one can construct objects y^i and $\partial/\partial y^i$, having the commutation relations of a set of classical coordinates and their derivatives, as ‘functions’ of x^i, e_a, h and a free parameter α . The ‘functions’ are in fact power series in h . The zero degree term for y^i is essentially fixed by (6.4). We stress in advance that the manipulations reported below are heuristic and formal; these manipulations will have to be justified in the proper representation theory framework.

The algebra of observables \mathcal{D}_h is a deformed Heisenberg algebra of dimension 3 (h is the deformation parameter). In fact, clearly the commutation relations between the x^i are such that

$$[x^i, x^j] = O(h).$$

Moreover, the change of generators (4.43) is independent of α (the dependence in θ_i^a and λ_a cancel), invertible, and gives

$$[\partial_i, x^j] = \delta_j^i + O(h), \quad [\partial_i, \partial_j] = O(h).$$

It is known (see e.g. [26] and references therein) that for any deformation \mathcal{D}_h of a Heisenberg algebra \mathcal{D} there exists an isomorphism $\mathcal{D}_h \leftrightarrow \mathcal{D}[[h]]$ of algebras over $\mathbb{C}[[h]]$. Applied to the present case, this implies that there exists a formal realization of the canonical generators of \mathcal{D} in terms of x^i, ∂_j , namely a set of formal power series in h

$$y^i = f^i(x, \partial, h) = x^i + O(h), \quad \frac{\partial}{\partial y^i} = g_i(x, \partial, h) = \partial_i + O(h),$$

with coefficient equal to polynomials in x^i, ∂_j and reducing to the identity for $h = 0$, such that the objects $y^i, \partial/\partial y^i$ fulfill the canonical commutation relations

$$[y^i, y^j] = 0, \quad \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right] = 0, \quad \left[\frac{\partial}{\partial y^i}, y^j \right] = \delta_j^i. \tag{6.7}$$

when $h = 0$ the above transformation is the identity, so also for $h \neq 0$ one can formally invert it to obtain a realization of the ‘deformed generators’ x^i, ∂_i in the form of formal power series in h with coefficients equal to polynomials in $y^i, \partial/\partial y^i$. (As a consequence, also e_a can be expressed in the same form; one can show the same also for Λ .) The above transformation is not unique. It is defined upto an infinitesimal inner automorphism of the algebra

$$x^i \rightarrow u x^i u^{-1}, \quad \partial_i \rightarrow u \partial_i u^{-1}, \tag{6.8}$$

where $u = 1 + O(h) \in \mathcal{D}_h[[h]]$. Since the commutation relations (6.7) are preserved upon rescaling and translations of the y^i by some constants $\alpha, c^i \in \mathbb{C}$, $y^i \rightarrow c^i + \alpha y^i$, if we relax the condition that the transformation be the identity at zero order in h a larger family of realizations will be given by

$$y^i = \alpha^{-1} [f^i(x, \partial, h) - c^i] = \alpha^{-1} [x^i - c^i + O(h)], \tag{6.9}$$

$$\frac{\partial}{\partial y^i} = \alpha g_i(x, \partial, h) = \alpha [\partial_i + O(h)]. \tag{6.10}$$

Next we shall fix the parameters α, c^i through some basic requirements on the commutative limit. As for the remaining indeterminacy (6.8), it should be fixed by geometrical requirements for finite (though small) h , e.g. the ones suggested at the end of the section.

In the first identification, namely (6.2), we have picked $c^i = 0$ and α finite, e.g. $\alpha = 1$; consequently $y^i = \lim_{h \rightarrow 0} x^i$. In the second identification we choose $c^\pm = 0$ and α a suitable infinitesimal in h so that the limit (6.4) is fulfilled; consequently, y^\pm will be recovered as the limit $\lim_{h \rightarrow 0} (x^\pm/\alpha)$. In order that formulae (6.9) and (6.10) contain a unique expansion parameter α , instead of two α, h , we choose the infinitesimal as $\alpha = O(h^{1/p})$, with some $p \in \mathbb{N}$ to be determined. We also choose $c^0 \neq 0$ in order that x^0 does not become singular in the commutative limit. If for simplicity we normalize c^0 to 1, we will find $x_{(0)}^0 = 1$. Together with (6.4), this implies $r_{(0)} = 1$. Note that any polynomial (or power series) of the x^i, r with finite coefficients has a constant (i.e. independent of y^i) commutative limit, because $\alpha \rightarrow 0$. We shall formally identify the three y^i thus obtained

as a set of (local) classical coordinates on the limit manifold, and $\partial/\partial y^i$ as the derivatives with respect to y^i . Thus the limit manifold will have dimension three. In the commutative limit

$$r \rightarrow 1, \quad (\alpha r)^{-1} dx^i \rightarrow dy^i, \tag{6.11}$$

and the line element (6.2) becomes

$$ds^2 \rightarrow \sum_i dy^i dy^{-i}. \tag{6.12}$$

This yields a vanishing Riemannian curvature and is consistent with the vanishing (for all q) linear curvature computed in Section 5. Assuming a trivial topology, the limit Riemannian manifold will be thus Euclidean space, and y^i will be (complex) cartesian coordinate on \mathbb{R}^3 .

From $r_{(0)} = x_{(0)} = \Lambda_{(0)} = 1$ one also easily finds

$$\lim_{\alpha \rightarrow 0} \alpha^{-1} q \Lambda e_a^i = \delta_a^i, \quad \lim_{\alpha \rightarrow 0} \alpha \Lambda^{-1} \theta_i^a = \delta_i^a,$$

whence

$$\lim_{\alpha \rightarrow 0} e_a = \lim_{\alpha \rightarrow 0} q \Lambda e_a^i \partial_i = \lim_{\alpha \rightarrow 0} \alpha^{-1} q \Lambda e_a^i \frac{\partial}{\partial y^i} = \frac{\partial}{\partial y^a}, \tag{6.13}$$

$$\lim_{\alpha \rightarrow 0} \theta^a = \lim_{\alpha \rightarrow 0} \Lambda^{-1} \theta_i^a dx^i = \lim_{\alpha \rightarrow 0} \Lambda^{-1} \theta_i^a \alpha \alpha^{-1} dx^i = dx^a. \tag{6.14}$$

This is consistent with the metric, stating that the set $\{\theta^a\}$ (and, in the dual formulation, the set $\{e_a\}$) is orthonormal. Let us determine possible values for α . Plugging (6.9) in the commutation relations (3.10), it is easy to see that at lowest order in α the third relation can be satisfied only if $p \geq 3$. We shall later exhibit a transformation (6.10) based indeed on $\alpha = O(\hbar^{1/3})$.

To understand the physical meaning of α one would have to consider the $*$ -representations of \mathcal{D} . We shall see elsewhere that it is related to the Planck length. Moreover in the commutative limit y^0 will ‘inherit’ from the $*$ -representation studied previously [13] a spectrum dense in the whole \mathbb{R} , in agreement with the fact that y^0 becomes a real cartesian coordinate. Similarly, $y_{\perp}^2 := y^+ y^-$ ‘inherits’ a spectrum dense in the whole \mathbb{R}^+ , in agreement with the fact that it becomes the square distance from the y^0 axis.

So far we have left the realization (6.9) and (6.10) still largely undetermined by the inner automorphism freedom (6.8). We may restrict this freedom by imposing further requirements on x^i, e_a , so that the physical meaning of the latter at the representation theoretic level become more manifest; for instance, since they commute and are real, we may require that at all orders in $\alpha x^0, r$ depends only on \bar{y} , so that their meaning of position observables be manifest. This will be discussed in detail elsewhere. Once determined, the realization (6.9) and (6.10) will be an essential ingredient of the *correspondence principle* between the deformed and undeformed theory for finite \hbar . An example, a formal realization of the algebra fulfilling all previous requirements and such that the classical involution $(y^i)^* = y^{-i}$ realizes also the involution $*$ of \mathcal{A} is

$$x^0 = \exp\left(\alpha y^0 - \frac{\alpha^2}{2}\right), \tag{6.15}$$

$$\Lambda = \exp\left(-\alpha^2 \frac{\partial}{\partial y^0}\right), \tag{6.16}$$

$$x^+ = \sqrt{\frac{\exp(\alpha^2 y^+ y^- - 1)}{q^2(q+1)}} \exp(\alpha y^0 + i\varphi) \exp\left(-\alpha^2 \frac{\partial}{\partial y^0} + 2\alpha \frac{\partial}{\partial (y^+ y^-)}\right), \tag{6.17}$$

$$x^- = (x^+)^* \sqrt{q} \tag{6.18}$$

with $e^{\alpha^3} = q$ and $e^{2i\varphi} = y^+ / y^-$. Its derivation will be given elsewhere. As a consequence

$$r^2 = e^{-2\alpha^3 + \alpha^2 y^+ y^- + 2\alpha y^0}, \quad \left(\frac{x^0}{r}\right)^2 = e^{\alpha^3 - \alpha^2 y^+ y^-}. \tag{6.19}$$

Thus, the surface $r = \text{const.}$ are paraboloids with axis y^0 , the surfaces $x^0 = \text{const.}$ are planes perpendicular to y^0 , the surface $x^0/r = \text{const.}$ are cylinders with axis y^0 , and the lines $x^0 = \text{const.}, r = \text{const.}$ are circles perpendicular to and with center on the axis y^0 . The exponential relation between x^0 and y^0 is analogous to the one found [3] for a one-dimensional q -deformed model. From the spectrum of x^0 [13] y^0 will inherit equidistant eigenvalues of both signs, suggesting a ‘uniform’ structure of space. This will solve the problems [13] arising from the physical identification of x^i as cartesian coordinates. Summing up, the resulting geometry is now flat \mathbb{R}^3 , instead of the $S^2 \times \mathbb{R}$ found in the first identification (the one with finite α). x^0, r may be adopted together with φ as global curvilinear coordinates, but only for finite $\alpha \approx h^{1/p}$, when they are to be identified with the coordinates (6.15) and (6.19); for $\alpha \rightarrow 0$ they become singular, and coordinates y^i may be used at their place.

7. The involution and the real calculus

Formula (4.3) gives an involution of $\Omega^1(\mathcal{A}) \oplus \bar{\Omega}^1(\mathcal{A})$. Unfortunately, the latter has rank 6 as a \mathcal{A} -bimodule, instead of 3, and is not generated from \mathcal{A} through the action of a real differential.

The problem of constructing a differential calculus of rank 3 closed under involution has been considered by Ogievetsky and Zumino [25], who expressed $\bar{\xi}^i$ as functions not only of $x^i \xi^i$, but also of suitably q -deformed derivations ∂_i . A much more economical construction is the following. Note that one can define an algebra isomorphism $\varphi : \Omega^*(\mathcal{A}) \rightarrow \bar{\Omega}^*(\mathcal{A})$ acting as the identity on \mathcal{A} and on the 1-forms through

$$\varphi(\theta^a) = \bar{\theta}^a, \tag{7.1}$$

since the commutation relations among the $\theta^{a'}$ ’s and the $\bar{\theta}^{a'}$ ’s are the same. Hence $\star = * \circ \varphi$ is an involution of $\Omega^*(\mathcal{A}a)$ acting as

$$(\theta^a)^* = \theta^b g b a. \tag{7.2}$$

This implies that $(\xi^i)^*$ can be expressed as combinations of ξ^j with coefficients in (the extended) \mathcal{A} . One can easily check that \star has the correct classical limit provided the commutative limit fulfills (6.4). Nevertheless, since $\varphi(d) \neq \bar{d}$, of course d is not real under the involution.

It would be natural to define a real exterior derivative by

$$d_r := (d + \bar{d})$$

and a rank 3 \mathcal{A} -bimodule $\Omega_r^1(\mathcal{A}) \in \oplus \bar{\Omega}^1(\mathcal{A})$ closed under \star from the generators

$$\xi_r^i = \xi^i + \bar{\xi}^i.$$

But this is impossible since ξ_r^i 's do not close commutation relations with the x^j 's, as evident from (4.1) and (4.2). To generate the exterior algebra through a real differential we need to double either the number of the coordinates or the number of 1-forms.

In general, consider an algebra \mathcal{A} with involution over which there are two differential calculi $(\Omega^*(\mathcal{A}), d)$ and $(\bar{\Omega}^*(\mathcal{A}), \bar{d})$ neither of which is necessarily real. Consider the product algebra $\tilde{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$ and over $\tilde{\mathcal{A}}$ the differential calculus

$$\tilde{\Omega}^*(\tilde{\mathcal{A}}) = \Omega^*(\mathcal{A}) \times \bar{\Omega}^*(\mathcal{A}). \tag{7.3}$$

It has a natural differential given by $\tilde{d} = (d, \bar{d})$. The embedding

$$\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$$

given $f \mapsto (f, f)$ is well defined and compatible with the involution

$$(f, g)^* = (g^*, f^*) \tag{7.4}$$

on $\tilde{\mathcal{A}}$. Suppose there exists a frame θ^a for $\Omega^*(\mathcal{A})$ and a frame $\bar{\theta}^a$ for $\bar{\Omega}^*(\mathcal{A})$ and a relation of the form (4.54) between them which extends the involution (7.4). We can define a real module $\Omega_r^1(\mathcal{A})$ to be the \mathcal{A} -bimodule generated by the frame

$$\theta_r^a = (\theta^a, \bar{\theta}^b g_{ba}).$$

This does not necessarily contain however the image of $d_r = (d, \bar{d})$. In fact, in the case $\mathcal{A} = \mathbb{R}_q^3$ if we require this to be the case then $\Omega_r^1(\mathcal{A})$ is an $\tilde{\mathcal{A}}$ -bimodule and $\Omega_r^*(\mathcal{A}) = \tilde{\Omega}_*(\mathcal{A})$.

Alternatively one can consider a larger ‘function’ algebra containing two copies of \mathcal{A} , whose generators we denote by x^i, \bar{x}^i , with cross commutation relations of the form

$$x^i \bar{x}^j = \hat{R}^{\pm 1} i_{j h k} \bar{x}^h x^k. \tag{7.5}$$

We can define an involution by setting

$$(x^i)^* = \bar{x}^j g_{ji} \tag{7.6}$$

and introduce an $SO_q(3)$ -covariant differential calculus $\Omega(\tilde{\mathcal{A}})$ with a real differential d_r by setting

$$d_r x^i = \xi^i, \quad d_r \bar{x}^i = \bar{\xi}^i, \tag{7.7}$$

with the commutation relations

$$\begin{aligned} x^i \xi^j &= q \hat{R}_{hk}^{ij} \xi^h x^k, & \bar{x}^i \bar{\xi}^j &= q^{-1} \hat{R}_{hk}^{-1ij} \bar{\xi}^h \bar{x}^k, & x^i \bar{\xi}^j &= 1^{-1} \hat{R}_{hk}^{\pm 1ij} \bar{\xi}^h x^k, \\ \bar{x}^i \xi^j &= q \hat{R}_{hk}^{\mp 1ij} \xi^h \bar{x}^k. \end{aligned} \tag{7.8}$$

(In the last two equations we have picked a specific normalization.) These relations are compatible with (7.6) and (4.44) and $(d_r f)^* = d_r f^*$. The commutation relations among the ξ^i and the $\bar{\xi}^i$ are the same relations (4.6) and (4.44) found in Section 4 for the analogous elements in the conjugate calculi, whereas the cross relations are

$$\xi^i \bar{\xi}^j - q^{-1} \hat{R}_{hk}^{\pm 1ij} \bar{\xi}^h \xi^k. \tag{7.9}$$

Thus one ends up with a real calculus with six independent coordinates and six independent 1-forms. But one can check that in this scheme there is no frame.

Finally, if we just let both the conjugate differentials d, \bar{d} act on one copy of \mathcal{A} we generate an \mathcal{A} -bimodule of rank 6 closed under the involution $*$. An enlarged exterior algebra is generated by $\xi^i, \bar{\xi}^i$ and the generators of \mathcal{A} ; we need to add to the commutation relations (4.1),(4.6),(4.2) and (4.44) compatible commutation relations between ξ^i and $\bar{\xi}^j$. It is easy to show that the latter can only be of the form $\xi^i \bar{\xi}^j = \gamma \hat{R}_{hk}^{-1ij} \bar{\xi}^h \xi^k$. This is compatible also with this algebra being a $\Omega^*(\mathcal{A})$ and a $\bar{\Omega}^*(\mathcal{A})$ bimodule. If we extend d, \bar{d} by requiring that

$$d\bar{\xi}^i = 0, \quad \bar{d}\xi^i = 0, \tag{7.10}$$

γ will be fixed to be $-q^{-1}$, and we find again (7.9). It is immediate to show that $\theta, \bar{\theta}$ are the Dirac operators of d, \bar{d} also in the enlarged algebra. A direct and lengthy calculation (which we have performed using the program for symbolical computations REDUCE) shows that the $\theta_i^a, \bar{\theta}_j^a$ introduced in Section 4 satisfy commutation relations

$$\hat{R}_{cd}^{ab} \bar{\theta}_j^d \theta_i^c = \theta_i^b \bar{\theta}_k^a \hat{R}_{ij}^{kl}. \tag{7.11}$$

As a consequence of (7.9) and (7.11), in the enlarged exterior algebra we find

$$\theta^a \bar{\theta}^b = -q \hat{R}_{cd}^{-1ab} \bar{\theta}^c \theta^d. \tag{7.12}$$

We can also readily extend the generalized permutation σ and the metric g of Section 5 to $\Omega^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A})$. For simplicity we consider an ansatz where σ does not mix the θ^a 's and the $\bar{\theta}^a$'s:

$$\sigma(\theta^a \otimes \bar{\theta}^b) = V^{abcd} \bar{\theta}^c \otimes \theta^d, \quad \sigma(\bar{\theta}^a \otimes \theta^b) = \bar{V}_{cd}^{ab} \theta^c \otimes \bar{\theta}^d. \tag{7.13}$$

Imposing conditions (2.14), we find the numerical matrices V, \bar{V} :

$$V = q \hat{R}^{-1}, \quad \bar{V} = q^{-1} \hat{R}. \tag{7.14}$$

Together with (5.1) and (5.12), these relations completely define σ . If we set

$$g(\theta^a \otimes \bar{\theta}^b) = g^{ab}, \tag{7.15}$$

$$g(\bar{\theta}^a \otimes \theta^b) = \bar{g}^{ab}, \tag{7.16}$$

the metric compatibility conditions (2.39),

$$d \circ g = g_{23} \circ D_2, \tag{7.17}$$

$$\bar{d} \circ g = g_{23} \circ \bar{D}^2 \tag{7.18}$$

on $\Omega^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A})$ and $\bar{\Omega}^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$ read

$$\bar{V}_{df}^{ae} g^{fg} S_{eg}^{cb} = g^{ac} \delta_d^b, \quad \bar{V}_{df}^{ae} \bar{g}^{fg} S_{eg}^{cb} = \bar{g}^{ac} \delta_d^b. \tag{7.19}$$

They can be satisfied only if the numerical matrices g^{ab}, \bar{g}^{ab} are proportional to the matrix (3.11) and we restrict (5.4) to the choice

$$S = q \hat{R}, \quad \bar{S} = (q \hat{R})^{-1}. \tag{7.20}$$

Note that in (7.19) no conformal factor appears. The connections defined by (5.7) and (5.16), are now automatically extended to the exterior algebra of rank 6, and it is easy to check that also on this larger domain their linear curvatures are zero. Moreover, they are still related by (5.17). We conclude by noting that the element

$$ds^2 := \theta^a \otimes \bar{\theta}^b g_{ab} + \text{h.c.}$$

of $\Omega^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A}) \otimes \bar{\Omega}^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$ is real and annihilated by the action of both D_2, \bar{D}_2 (in other words it is 'invariant under parallel transport'). Therefore it might be considered as a candidate for the square displacement.

8. Conclusions

The fundamental open problem of the noncommutative theory of gravity concerns of course the relation it might have to a future quantum theory of gravity either directly or via the theory of strings and membranes. But there are more immediate technical problems which have not received a satisfactory answer. The most important ones concern the definition of the curvature. It is not certain that the ordinary definition of curvature taken directly from differential geometry is the quantity which is most useful in the noncommutative theory. The main interest of curvature in the case of a smooth manifold definition of space–time is the fact that it is local. We have defined Riemann curvature as a map Curv which takes $\Omega^1(\mathcal{C}(V))$ into $\Omega^2(\mathcal{C}(V)) \oplus \mathcal{C}(V)\Omega^1(\mathcal{C}(V))$. If $\xi \in \Omega^+(\mathcal{C}(V))$ then $\text{Curv}(\xi)$ at a given point depends only on the value of ξ at that point. This can be expressed as a bilinearity condition; the above map is a $\mathcal{C}(V)$ -bimodule map. If $f \in \mathcal{C}(V)$ then

$$f\text{Curv}(\xi) = \text{Curv}(f\xi), \quad \text{Curv}(\xi f) = \text{Curv}(\xi) f. \tag{8.1}$$

One would like to insure also that one is dealing with the noncommutative version of a real manifold. This reality condition exchanges left and right linearity and adds weight to the argument that a bilinear curvature map is necessary. In the noncommutative case bilinearity is therefore the natural and only possible expression of locality. It has not yet been possible to enforce it in a satisfactory manner [8].

We have argued here in favor of one possible extension to noncommutative geometry of the definition of a linear connection which relies essentially on the above expression of reality and locality. Alternative extensions have however been given. In fact one definition has been proposed [5] which is indeed local in the sense we have defined locality but which is valid only in the noncommutative case and becomes singular in the commutative limit. The main difference with alternative definitions and the one we use lies in the fact that the locality condition is relaxed and the Leibniz rule is applied only from one side; the module of 1-forms is considered only as a left (or right) module. This means in fact that gravity is considered as a Yang–Mills field with the general linear group of some q -deformation of it, as structure group. One of the authors has given elsewhere [19] a partial list of these alternative proposals. We mention here only three which are especially relevant to the example of quantum Euclidean space. Ref. [15] is the closest to ours. A quantum group ($SL_q(N)$, $O_q(N)$ or $Sp_q(N)$) function algebra, rather than a quantum space function algebra, is chosen as algebra \mathcal{A} . General definitions of a metric, covariant derivative, torsion, curvature, metric-compatible covariant derivative on \mathcal{A} are given; in particular, the quantum group invariant/bicovariant ones are determined, the definitions differ from ours essentially in that \mathcal{A} -bilinearity is weakened to a left-(or right-) linearity. The existence of frames is not investigated. In Ref. [2] a q -Poincaré group function algebra is chosen as an algebra \mathcal{A} ; the definition of differential calculus, linear connection, curvature, mimic the group-geometric ones on the ordinary Poincaré group manifold. The underlying Hopf algebra is triangular, which essentially means $\hat{R}^2 = 1$ (in this respect the situation is simpler than the one considered in the present work). The existence of metric, torsion and frames is not investigated. The q -Minkowski space, its differential calculus, linear connection, curvature, are meant to be derived by projecting out the q -Lorentz group. Other authors [9,22] have abandoned the definition of a connection as a covariant derivative and proposed a q -deformed version of a principle bundle to define a noncommutative ‘Riemannian geometry’, using for example the universal calculus. It is interesting to note that according to the definition we use here the unique linear connection associated with any universal calculus is necessarily trivial [7]; the covariant derivative coincides with the ordinary exterior derivative.

Finally, we would like to compare the results found here for the quantum Euclidean space with the result found in [6] for the Manin quantum plane. We start by noting that all commutation relations in both cases are homogeneous separately in x^i and ξ^i . A Dirac operator θ (2.1) for a quantum group covariant differential calculus must be of degree 1 in ξ^i , degree-1 in x^j and quantum group invariant. There can be no such object for the Manin plane because, to mimic the construction (4.13), we would have to use the isotropic tensor ε_{ij} of $SL_q(2)$, the q -deformed ε -tensor in the place of the isotropic tensor g_{ij} of $SO_q(3)$, but the analog $x^i x^j \varepsilon_{ij}$ of r^2 vanishes, by the q -commutation relations of the Manin plane. The absence of such a θ prevents the construction of a torsion-free linear connection by means of formula (2.22) which, by means of Eq. (2.26), could eventually yield a bilinear zero curvature. In Ref. [6] a Stehbein was constructed using the inverse of both Manin plane coordinates and no dilatator, whereas we have needed here the inverse of one coordinate x^0 and the introduction of the dilatator. Using the Stehbein, also a metric has been found.

However, it is not a quantum group covariant metric, as found here. One could define a $SL_q(2)$ -covariant metric g_0 on the Manin plane by introducing a dilatator Λ , in the same way as in Eqs. (4.16) and (4.25) and by setting $g_0(\xi^i \otimes \xi^j) = \Lambda^{-3} \varepsilon^{ij}$ (see Eq. (5.6)). This would be a symplectic form in the limit $q = 1$. An analog of Eq. (7.19), but in the ξ^i basis, would hold.

Acknowledgements

One of the authors (JM) would like to thank J. Wess for his hospitality and the Max-Planck-Institut für Physik in München for financial support.

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